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Polyakov lines in Yang-Mills matrix models

Peter Austing and John F. Wheater

Department of Physics, University of Oxford
Theoretical Physics,
1 Keble Road, Oxford OX1 3NP, UK
E-mail: p.austing@physics.ox.ac.uk, j.wheater@physics.ox.ac.uk

Graziano Vernizzi

Service de Physique Théorique, CEA/DSM/SPhT Saclay
Unité de recherche associée au CNRS
F-91191 Gif-sur-Yvette Cédex, France
E-mail: vernizzi@spht.saclay.cea.fr

Abstract: We study the Polyakov line in Yang-Mills matrix models, which include the IKKT model of IIB string theory. For the gauge group SU(2) we give the exact formulae in the form of integral representations which are convenient for finding the asymptotic behaviour. For the SU(N) bosonic models we prove upper bounds which decay as a power law at large momentum p. We argue that these capture the full asymptotic behaviour. We also indicate how to extend the results to some correlation functions of Polyakov lines.

Keywords: Nonperturbative Effects, M(atrix) Theories, Matrix Models.

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Contents

1. Introduction

The proposal [1] for a matrix model describing IIB string theory has triggered considerable interest in Yang-Mills matrix models. The models are defined in section 2, and it is the $D = 10$ dimensional maximally supersymmetric version which is proposed as the model of string theory. When the gauge group is SU(2) it has been known for some time how to compute the supersymmetric integrals [2]–[7]. It was believed that the bosonic integrals (in which the fermionic degrees of freedom are omitted) would diverge because of the flat directions in which the matrices all commute, but the authors of [8] computed them analytically for SU(2) and numerically for some other groups, and realised they can converge as long as D is big enough. We established analytically the convergence criteria for the bosonic SU(N) integrals in [9] and for the other gauge groups and supersymmetric integrals in [10], confirming that the partition function does exist in the bosonic case when D is big enough (except for SU(2) and SU(3), D need only be bigger than 2), and that the supersymmetric models exist when $D = 4$, 6 and 10. For further details, see also [11].

For SU($N > 2$) it is not known how to compute the integrals exactly, but the authors of [12] used the supersymmetry to deform the partition function into a cohomological theory in which the integrals can be done. Although the relation of this model to the original Yang-Mills model remains unproven, the resulting numbers have been confirmed by very careful numerical calculation for small values of N [13, 8], and support a conjecture [14] based on D brane dynamics. The method has also been extended to the other simple groups, and checked numerically for small rank [15, 16]. Unfortunately, it cannot be extended to observables such as the Polyakov line which break supersymmetry.

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Another important line of research has been into overcoming the considerable difficulties in obtaining numerical results for larger values of N in order to gain understanding of the large-N limit. Various simulations of the bosonic SU(N) model have been reported [17]–[21], and the Polyakov line and Wilson loop have been studied up to $N = 768$ in [22]. For the supersymmetric model with $D = 4$, the pfaffian is positive, and these models have been studied up to $N = 48$ [23, 24, 18], and with particular reference to Polyakov lines and Wilson loops [25]. For $D = 6$ and $D = 10$, the situation is much more difficult since the pfaffian is complex. Various ways of dealing with this have been tried [26]–[29]. Meanwhile, supersymmetric random surface, and geometrical approaches have been investigated in [30, 31], as have mean field theory and $1/D$ expansions in [32]–[37].

The purpose of this paper is to obtain rigorous information about the large p behaviour of the Polyakov line for SU(N) with finite N. The scene is set in section 2. In section 3 we derive the exact results for the bosonic and supersymmetric SU(2) models; this is not new but our derivation gives an integral representation which enables the asymptotic behaviour to be calculated very easily. In section 4 we derive upper bounds on the large p behaviour for the Polyakov line in the bosonic theory. These decay polynomially, and as we shall indicate, there is good reason to believe that the upper bounds capture the true behaviour. We also indicate how to extend the results to some correlation functions of Polyakov lines. Section 5 contains a brief discussion.

2. Definition of the models

The Yang-Mills matrix integral partition function, which is obtained by dimensionally reducing the euclidean SSYM action from D down to zero dimensions, is given by

$$\mathcal{Z}_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \prod_{\alpha=1}^{\mathcal{N}} d\psi_\alpha \exp\left(\frac{1}{4}\sum_{\mu,\nu} \mathrm{Tr}\ [X_\mu, X_\nu]^2 + \frac{1}{2}\mathrm{Tr}\, \psi_\alpha[\Gamma^\mu_{\alpha\beta}X_\mu, \psi_\beta]\right), \tag{2.1}$$

where we adopt the summation convention for repeated indices. The traceless hermitian matrix fields X_μ and ψ_α (respectively bosonic and fermionic) are in the Lie algebra $\mathcal{G}$ of the (compact semi-simple) gauge group G and can be written

$$X_\mu = \sum_{a=1}^{g} X^a_\mu t^a\,, \qquad \psi_\alpha = \sum_{a=1}^{g} \psi^a_\alpha t^a\,, \tag{2.2}$$

where $\{t^a, a = 1, \ldots, g\}$ are the generators in the fundamental representation. The $\Gamma^\mu_{\alpha\beta}$ are ordinary gamma matrices for D euclidean dimensions. The model possesses a gauge symmetry

$$X_\mu \to U^\dagger X_\mu U\,, \qquad \psi_\alpha \to U^\dagger \psi_\alpha U\,, \qquad U \in G\,. \tag{2.3}$$

and an SO(D) symmetry inherited from the original D-dimensional euclidean symmetry of the SSYM. Although the motivation discussed above leads to a study of the $D = 10$, SU(N) supersymmetric integral, it is useful and illuminating to study several different versions of the model. Firstly by suppressing the fermions we get the bosonic integrals which we will denote by $\mathcal{N} = 0$ (ie there are no super-charges) [8]. Secondly the supersymmetric integrals can be written for $D = 3$, 4, 6, and 10, having $\mathcal{N} = 2(D-2)$ super-charges. In principle one can integrate out the fermions to obtain

$$\mathcal{Z}_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \, \mathcal{P}_{D,G}(X_\mu) \exp\left(-S_D(X)\right), \tag{2.4}$$

where

$$S_D(X) = -\frac{1}{4} \sum_{\mu,\nu=1}^{D} \mathrm{Tr}\, [X_\mu, X_\nu]^2 \tag{2.5}$$

and the pfaffian $\mathcal{P}_{D,G}$ is a homogeneous polynomial of degree $\frac{1}{2}\mathcal{N}g$. In this paper we will be concerned with Polyakov line correlation functions of the form

$$\begin{aligned} L(P) &= \langle \mathrm{Tr}\, \exp(iP^\sigma X_\sigma) \rangle \\ &= \int \prod_{\mu=1}^{D} dX_\mu \, \mathrm{Tr}\, \exp(iP^\sigma X_\sigma) \, \mathcal{P}_{D,G}(X_\mu) \exp\left(\frac{1}{4}\sum_{\mu,\nu} \mathrm{Tr}\, [X_\mu, X_\nu]^2\right). \end{aligned} \tag{2.6}$$

Using the SO(D) symmetry we can always consider $P^\sigma = (p, 0, \ldots)$ and denote the corresponding Polyakov line by $L(p)$.

It is convenient to express the Lie algebra $\mathcal{G}$ using the Cartan-Weyl basis

$$\{H^i, E^\alpha\}, \tag{2.7}$$

where i runs from 1 to the rank r and α denotes a root. In this basis

$$\left[H^i, H^j\right] = 0\,, \qquad \left[H^i, E^\alpha\right] = \alpha^i E^\alpha \tag{2.8}$$

and

$$\begin{aligned} \left[E^\alpha, E^\beta\right] &= N_{\alpha\beta} E^{\alpha+\beta} && \text{if } \alpha+\beta \text{ is a root} \\ &= 2|\alpha|^{-2}\, \alpha \cdot H && \text{if } \alpha = -\beta \\ &= 0 && \text{otherwise}\,. \end{aligned} \tag{2.9}$$

Here $E^{-\alpha} = (E^\alpha)^\dagger$, and the normalisation is chosen such that

$$\mathrm{Tr}\, H^i H^j = \delta^{ij}\,, \qquad \mathrm{Tr}\, E^\alpha E^\beta = 2|\alpha|^{-2}\, \delta^{\alpha+\beta}\,, \qquad \mathrm{Tr}\, H^i E^\alpha = 0\,. \tag{2.10}$$

We can expand any matrix in the Lie Algebra as

$$M = \widetilde{M} \cdot H + \sum_\alpha \overline{M}^\alpha E^\alpha \tag{2.11}$$

with $\overline{M}^{-\alpha} = (\overline{M}^\alpha)^*$ (but for readability's sake we will only use this expansion when necessary). We will denote the set of all roots by Δ, the set of positive roots by Δ_+, and the set of simple positive roots by Δ_+^*; in addition we will denote a simple root by s, retaining α for a generic root. The dual weights ω_s satisfy

$$2\frac{s \cdot \omega_{s'}}{|s|^2} = \delta_{ss'}\,. \tag{2.12}$$

The above definitions (2.7)–(2.12) fix the normalisation of the long root to be $\sqrt{2}$. In the case of su(N), all roots are of equal magnitude $\sqrt{2}$.

Since the integrand and measure are gauge invariant, we can always make a gauge transformation (2.3) to move X_D into the Cartan subalgebra

$$X_D = \lambda_D \cdot H \tag{2.13}$$

and reduce the integral over X_D to an integral over its Cartan modes [15]

$$\prod_{a=1}^{g} dX_D^a \to \Omega_G \left(\prod_{i=1}^{l} d\lambda_D^i \right) \Delta_G^2(\lambda_D) \,, \tag{2.14}$$

where Ω_G is the volume of G and the Weyl measure, given by

$$\Delta_G^2(\lambda_D) = \prod_{\alpha \in \Delta_+} (\lambda_D \cdot \alpha)^2 \ , \tag{2.15}$$

is the generalisation from SU(N) of the Vandermonde determinant factors.

We will consider mainly the integral (2.6) without fermions so that $\mathcal{N} = 0$ and there is no pfaffian. In the Cartan representation the expression (2.6) takes the form

$$\begin{aligned} L(p) = & \int \left(\prod_{i=1}^{r} d\lambda_D^i \right) \Delta_G^2(\lambda_D) \prod_{k=1}^{D-1} dX_k \operatorname{Tr} \exp(ip\lambda_D \cdot H) \times \\ & \times \exp \left(- \sum_{\alpha>0} \frac{(\lambda_D \cdot \alpha)^2}{|\alpha|^2} \sum_{k=1}^{D-1} |\overline{X}_k^\alpha|^2 + \frac{1}{4} \sum_{j,k=1}^{D-1} \operatorname{Tr} \, [X_j, X_k]^2 \right) \end{aligned} \tag{2.16}$$

and since the weights in a given representation correspond to the eigenvalues of the Cartan generators, we have

$$\operatorname{Tr} \exp(ip\lambda_D \cdot H) = \sum_h \exp(ip\lambda_D \cdot h) \,, \tag{2.17}$$

where $\{h\}$ are the weights. In the case of su(N), every weight in the fundamental representation is a Weyl transformation of every other weight, so we can use Weyl invariance to write

$$L(p) = N \int d\lambda_D \Delta_G^2(\lambda_D) dX_k \, \exp \left(ip\lambda_D \cdot \omega_1 - \sum_{\alpha>0} \frac{(\lambda_D \cdot \alpha)^2}{|\alpha|^2} \sum_{k=1}^{D-1} |\overline{X}_k^\alpha|^2 - S_{D-1} \right), \tag{2.18}$$

since ω_1 is the first weight in the fundamental representation.

3. Exact calculations for su(2)

In the simplest case of su(2) it is possible to reduce $L(p)$ to a simple integral representation. This is because there are few enough fields to make good use of the rotation symmetry.[1] Several authors have used exact calculations of the su(2) partition function in various contexts [2]–[7], and integral representations of the eigenvalue density have been given in [39]. In principle one only need take the Fourier transform of the densities given in [39]

[1] The authors of [38] also managed to extend this to the integrals over the 3×3 real symmetric matrices.

to obtain the Polyakov line but it is hard to extract the asymptotic behaviour from these representations. The following calculations amount to an alternative representation for the eigenvalue densities from which the asymptotic behaviour of $L(p)$ can be easily determined.

The definitions in section 2 fix the normalisation of the generators, but to avoid any ambiguity in the definition (2.6) we give them explicitly here;

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

so that $H = \sigma_3$ and $E^{\pm 1} = 1/2(\sigma_1 \pm i\sigma_2)$.

3.1 Bosonic

There is only one positive root and (2.18) simplifies to

$$L(p) = 2\int d\lambda_D\, \lambda_D^2 \prod_{k=1}^{D-1} dX_k \, \exp\left(\frac{ip\lambda_D}{\sqrt{2}}\right) \exp\left(-\lambda_D^2 \sum_{k=1}^{D-1} \left|\overline{X}_k^1\right|^2 - S_{D-1}\right). \tag{3.2}$$

Setting

$$X_k = u_k\sigma_1 + v_k\sigma_2 + \xi_k\sigma_3\,, \quad k = 1, \ldots, d\,, \tag{3.3}$$

where $d = D - 1$, we get

$$L(p) = -2\frac{d^2}{dp^2}K(p)\,, \tag{3.4}$$

with

$$K(p) = \int d\lambda e^{ip\lambda} \prod_{k=1}^{D-1} d\xi_k du_k dv_k \exp\left(-\frac{1}{2}\sum_{i,j=1}^{d} \xi_i U_{ij} \xi_j - \frac{1}{4}V - \frac{\lambda^2}{2}W\right), \tag{3.5}$$

where

$$U_{ij} = (u^2 + v^2)\delta_{ij} - (u_i u_j + v_i v_j) \tag{3.6}$$
$$V = u^2 v^2 - (u \cdot v)^2 \tag{3.7}$$
$$W = u^2 + v^2\,. \tag{3.8}$$

Integrating out the d-dimensional vector ξ gives, up to a constant factor which we will fix later,

$$K(p) = \int d\lambda du\, dv\, (\det U)^{-\frac{1}{2}} \exp\left(\frac{ip\lambda}{\sqrt{2}} - \frac{V}{4} - \frac{\lambda^2 W}{2}\right) \tag{3.9}$$

which, noting that the determinant is $\det U = W^{d-2}V$, we rewrite as

$$\int dAdBd\lambda\, \frac{\exp(-\lambda^2 A/2 + ip\lambda/\sqrt{2} - B/4)}{A^{\frac{d-2}{2}} B^{\frac{1}{2}}} J(A, B)\,, \tag{3.10}$$

where

$$J = \int du\, dv\, \delta(A - W)\delta(B - V) \tag{3.11}$$
$$= \text{const}\, B^{\frac{d-3}{2}}\, \theta\left(\frac{A^2}{4} - B\right), \tag{3.12}$$

as one can check by using the rotation invariance. Then integrating out λ, and scaling A by p^2, we obtain

$$K(p) = \frac{(D-2)}{\Gamma\left(\frac{D}{4}-1\right)} p^{4-D} \int_0^\infty dB\, B^{\frac{d-4}{2}} e^{-B/4} \int_{\frac{2\sqrt{B}}{p^2}}^\infty dA\, A^{-\frac{d-1}{2}} e^{-\frac{1}{4A}}\,, \tag{3.13}$$

where we have fixed the constant factor by requiring $L(0) = 1$. Differentiating twice with respect to p, we obtain terms which vanish like a polynomial from differentiating p^{4-D} and terms which vanish exponentially from differentiating the integral. Then we find

$$L(p) \sim -2^{D-1}\sqrt{\pi}\frac{\Gamma(D-1)}{\Gamma\left(\frac{D}{4}-1\right)} p^{2-D} \tag{3.14}$$

as $p \to \infty$ for $D > 4$.[2]

3.2 Supersymmetric

For $D = 4$, 6 and 10, the pfaffian is given by [13]

$$\mathcal{P} = \left[\frac{8}{3}\mathrm{Tr}\, [X_\mu, X_\nu]\, [X_\nu, X_\rho]\, [X_\rho, X_\mu]\right]^{\frac{D-2}{2}} = \left[6\left(\lambda^2 V + \xi_i M_{ij}\xi_j\right)\right]^{\frac{D-2}{2}}\,, \tag{3.15}$$

where

$$M_{lm} = \delta_{lm}V - 2\left(v^2 u_l u_m - u\cdot v(v_l u_m + u_l v_m) + u^2 v_l v_m\right). \tag{3.16}$$

We note that M and U have the same eigenvectors. Specifically, choosing $d-2$ linearly independent vectors a each perpendicular to u and v, we have

$$Ua = Wa\,, \quad Ma = Va \tag{3.17}$$

$$Uu = v^2 u - u\cdot vv\,, \quad Mu = 0 \tag{3.18}$$

$$Uv = u^2 v - u\cdot vu\,, \quad Mv = 0\,. \tag{3.19}$$

The eigenvalues of U on the u-v space are q_{d-1} and q_d where $q_{d-1}q_d = V$.

Proceeding as in the bosonic case and working in the eigenvector basis of U and M gives

$$\begin{aligned} L(p) &= -\frac{d^2}{dp^2}\int d\lambda e^{ip\lambda/\sqrt{2}} d\xi du dv \left(\lambda^2 V + V\sum_{k=1}^{d-2}\xi_k^2\right)^{\frac{D-2}{2}} \times \\ &\quad \times \exp\left(-\frac{1}{2}W\sum_{k=1}^{d-2}\xi_k^2 - \frac{1}{2}q_{d-1}\xi_{d-1}^2 - \frac{1}{2}q_d\xi_d^2 - \frac{1}{4}V - \frac{1}{2}\lambda^2 W\right). \end{aligned} \tag{3.20}$$

As before, we can use the rotation symmetry to do the u and v integrals; the pfaffian terms are pulled down by differentiating with respect to W; and then ξ_i is integrated out to give

$$L(p) = \frac{d^2}{dp^2}\int d\lambda e^{ip\lambda/\sqrt{2}} \int_0^\infty dA \int_0^{A^2/4} dB B^{\frac{D-3}{2}} B^{\frac{D-4}{2}} e^{-B/4}\left(-\frac{\partial}{\partial A}\right)^{\frac{D-2}{2}} A^{\frac{3-D}{2}} e^{-\lambda^2 A/2} \tag{3.21}$$

[2]For su(2), the partition function diverges when $D \leq 4$.

from which the eigenvalue density $\rho(\lambda)$ can be read off. Integrating out λ gives

$$L(p) = -\mathcal{N}_D \frac{d^2}{dp^2} \int_0^\infty dA\, A^{2D-6} e^{-A^2/16} \left(-\frac{\partial}{\partial A}\right)^{\frac{D-4}{2}} A^{\frac{2-D}{2}} e^{-\frac{p^2}{4A}}\,, \tag{3.22}$$

where the overall constant factor

$$\mathcal{N}_D = \frac{2^{3(3-D)}(D-3)(D-2)\sqrt{\pi}}{\Gamma\left(\frac{D-3}{2}\right)^2} \tag{3.23}$$

is fixed by requiring $L(0) = 1$. Then, by the saddle point method,

$$L(p) \sim (-1)^{\frac{D}{2}+1} 2^{3-\frac{5D}{6}} \sqrt{\frac{\pi}{3}} \mathcal{N}_D\, p^{\frac{4D-12}{3}} \exp\left(-\frac{3}{16} 2^{\frac{2}{3}} p^{\frac{4}{3}}\right) \tag{3.24}$$

as $p \to \infty$. Note that the asymptotic behaviour is exponential and that the power law terms present in the bosonic case have disappeared. It is tempting to deduce that this is a consequence of the supersymmetry. However it is easy to check that replacing the $\frac{1}{2}(D-2)$ power in the expression for the pfaffian (3.15) by any positive integer has the same effect of suppressing the power law terms but does not in general come from a supersymmetric theory.[3] The quantity appearing in (3.15) is special in su(2) because it is proportional to a derivative of the action when expressed in appropriate variables; it is the fact that one of the integrals is then of a total derivative that suppresses the power law.

4. Upper bounds on bosonic large p behaviour

By using the convergence techniques of [9, 10] we can obtain an upper bound on the large p behaviour of the Polyakov line for su(N). We begin by illustrating the method on su(2) and then on su(3) before giving the general argument.

4.1 su(2)

Starting from (3.2) we integrate out λ_D, giving

$$L(p) = \frac{\sqrt{\pi}}{2} \int \prod_{k=1}^{D-1} dX_k\, Q^{-\frac{3}{2}} \left(2 - \frac{p^2}{2Q}\right) \exp\left(-\frac{p^2}{8Q} - S_{D-1}\right), \tag{4.1}$$

where

$$\begin{aligned} Q &= \sum_{k=1}^{D-1} \left|\overline{X}_k^1\right|^2, \\ S_{D-1} &= -\frac{1}{4} \sum_{j,k=1}^{D-1} \operatorname{Tr}\, [X_j, X_k]^2\,, \end{aligned} \tag{4.2}$$

[3] We thank Bergfinnur Durhuus for pointing this out to us.

so that a bound on $L(p)$ is given by

$$|L(p)| < \frac{\sqrt{\pi}}{2} \int \prod_{k=1}^{D-1} dX_k \, Q^{-\frac{3}{2}} \left(2 + \frac{p^2}{2Q}\right) \exp\left(-\frac{p^2}{8Q} - S_{D-1}\right). \tag{4.3}$$

In the region of integration which leads to power law behaviour, we must have $Q \to \infty$ but S_{D-1} bounded, so we will eventually need to examine the flat directions of the $(D-1)$-dimensional action S_{D-1}. First though, in any region in which $Q^{-1} > p^{-2+2\epsilon}$ with ϵ a small positive parameter, we have exponential behaviour of $L(p)$, so we restrict attention to the region $Q^{-1} < p^{-2+2\epsilon}$, giving the bound

$$|L(p)| < \frac{\sqrt{\pi}}{2} \int \prod_{k=1}^{D-1} dX_k \, p^{-3+3\epsilon} \left(2 + \frac{1}{2} p^{2\epsilon}\right) \exp\left(-\frac{p^2}{8Q} - S_{D-1}\right). \tag{4.4}$$

In order to consider the flat directions, we use the radial variable R defined by

$$X_k = R x_k \,, \qquad \sum_{k=1}^{D-1} \mathrm{Tr}\, x_k x_k = 1 \,. \tag{4.5}$$

Then certainly $Q < R^2$, so we have the bound

$$|L(p)| < p^{-3+3\epsilon} \left(2 + \frac{1}{2} p^{2\epsilon}\right) \frac{\sqrt{\pi}}{2} \int \prod_{k=1}^{D-1} dX_k \, \exp\left(-\frac{p^2}{8R^2} - R^4 \hat{S}_{D-1}\right) \tag{4.6}$$

where

$$\hat{S}_{D-1} = -\frac{1}{4} \sum_{j,k=1}^{D-1} \mathrm{Tr}\, [x_j, x_k]^2 \,. \tag{4.7}$$

To estimate the integral at large p we split the integration domain into two:

$$\begin{aligned} \mathcal{R}_1 &: \hat{S}_{D-1} \geq R^{\epsilon-4} \,, \\ \mathcal{R}_2 &: \hat{S}_{D-1} < R^{\epsilon-4} \,, \end{aligned} \tag{4.8}$$

where ϵ is a small positive constant. Since we know that the integrals are convergent the R integral in $\mathcal{R}_1$ may be evaluated by the saddlepoint method and we get a contribution to $L(p)$ which decays exponentially at large p. On the other hand the contribution from $\mathcal{R}_2$ cannot be treated in this way and generates power law behaviour. Bounds on the angular integrals in (4.6) have been computed in [9] and for su(2), they give

$$|L(p)| < p^{-3+3\epsilon} \left(2 + \frac{1}{2} p^{2\epsilon}\right) \frac{\sqrt{\pi}}{2} \int dR \, \exp\left(-\frac{p^2}{8R^2}\right) R^{-D+4+\epsilon'}, \tag{4.9}$$

where ϵ' is another arbitrarily small positive parameter, so that

$$|L(p)| < \mathrm{const}\, p^{2-D+\epsilon''} \tag{4.10}$$

as long as $D \geq 6$.

The difficulty in finding a lower bound is illustrated in this example. In (4.1), we have a positive term and a negative term, and each gives the same upper bound power law behaviour. One can apply the lower bound methods of [10], and find the same power law behaviour as lower bounds for each term (up to arbitrarily small parameters ϵ again). These terms originate from the Vandermonde factors and, since they each have the same power law behaviour, we cannot rule out cancellation between them.

4.2 su($N > 2$)

Completing the square in (2.18) and changing variables from λ to $\nu = Q^{\frac{1}{2}}\lambda$ gives,

$$L(p) = \int \frac{d\underline{\nu}}{\det(Q)^{\frac{1}{2}}} \prod_{k=1}^{D-1} dX_k \prod_{\alpha\in\Delta_+} \left(\left(Q^{-\frac{1}{2}}\nu + ipQ^{-1}\omega_1\right)\cdot\alpha\right)^2 \exp(-p^2\omega_1\cdot Q^{-1}\cdot\omega_1)\times$$
$$\times \exp\left(-\nu\cdot\nu + \sum_{j,k=1}^{D-1} \mathrm{Tr}\,[X_j, X_k]^2\right), \tag{4.11}$$

where

$$Q_{ij} = \sum_{\alpha>0} \frac{\alpha^i\alpha^j}{|\alpha|^2} T_\alpha$$
$$T_\alpha = \sum_{k=1}^{d} |X_k^\alpha|^2. \tag{4.12}$$

The integrand in (4.11) would be positive definite but for the Weyl determinant factor which we may bound above

$$\left|\prod_{\alpha\in\Delta_+} \left(\left(Q^{-\frac{1}{2}}\nu + ipQ^{-1}\omega_1\right)\cdot\alpha\right)^2\right| < \mathrm{const}\left[\sum_{\alpha\in\Delta_+}\left\{(\alpha^T Q^{-\frac{1}{2}}\nu)^2 + p^2(\alpha^T Q^{-1}\omega_1)^2\right\}\right]^{\frac{g-r}{2}}$$
$$= \mathrm{const}\left[\nu^T Q^{-1}\nu + p^2(\omega_1^T Q^{-2}\omega_1)\right]^{\frac{g-r}{2}}, \tag{4.13}$$

where we have used the relation

$$\sum_{\alpha\in\Delta_+} \alpha\alpha^T = \mathrm{const}\,1. \tag{4.14}$$

4.2.1 su(3)

Now we will temporarily restrict the algebra to su(3) and choose the basis $s_1 = (\sqrt{3/2}, -\sqrt{1/2})$, $s_2 = (0, \sqrt{2})$, in which, abreviating $T_i \equiv T_{\alpha_i}$,

$$Q^{-1} = \frac{1}{3(T_1T_2 + T_2T_3 + T_3T_1)} \begin{pmatrix} 4T_1 + T_2 + T_3 & -\sqrt{3}(T_2 - T_3) \\ -\sqrt{3}(T_2 - T_3) & 3(T_2 + T_3) \end{pmatrix} \tag{4.15}$$

and

$$\omega_1^T Q^{-1}\omega_1 = \frac{2}{9}\frac{4T_1 + T_2 + T_3}{(T_1T_2 + T_2T_3 + T_3T_1)}. \tag{4.16}$$

Since the T_a are positive, we can bound any matrix element by

$$x^T Q^{-1} y < \text{const}|x||y|\omega_1^T Q^{-1}\omega_1, \tag{4.17}$$

which applies in particular to the eigenvalues of Q^{-1}. Taking these bounds together with (4.13), we can bound the expression (4.11) for $L(p)$

$$|L(p)| < \int d\underline{\nu}(\omega_1^T Q^{-1}\omega_1)^{r/2} \prod_{k=1}^{D-1} dX_k\, (\omega_1^T Q^{-1}\omega_1)^{(g-r)/2} \sum_{n=0}^{(g-r)/2} a_n(\nu\cdot\nu)^{(g-r)/2-n} \times$$
$$\times (p^2\omega_1^T Q^{-1}\omega_1)^n \exp(-p^2\omega_1^T Q^{-1}\omega_1) \exp\left(-\nu\cdot\nu + \sum_{j,k=1}^{D-1} \mathrm{Tr}\,[X_j, X_k]^2\right),$$

where the a_n are the coefficients of the binomial expansion. We can now integrate out ν giving some new positive coefficients $\tilde{a}_n$,

$$|L(p)| < \int \prod_{k=1}^{D-1} dX_k \sum_{n=0}^{(g-r)/2} \tilde{a}_n p^{2n}(\omega_1^T Q^{-1}\omega_1)^{n+g/2} \exp\left(-p^2\omega_1^T Q^{-1}\omega_1 - S_{D-1}(X)\right). \tag{4.18}$$

We now split the integration region into two parts. In the region in which $\omega_1^T Q^{-1}\omega_1 \geq p^{-2+2\epsilon}$ with ϵ small but positive, we have exponential behaviour in p. We therefore restrict our attention to the region in which

$$\omega_1^T Q^{-1}\omega_1 < p^{-2+2\epsilon}. \tag{4.19}$$

Writing R as the radial variable $R^2 = \widetilde{X}_\mu^i \widetilde{X}_\mu^i + \overline{X}_\mu^\alpha \overline{X}_\mu^{-\alpha}$ (with summation over μ, i and α implied) we have the bound

$$\omega_1^T Q^{-1}\omega_1 > \frac{A}{R^2}, \tag{4.20}$$

where A is a constant, since the T_a are positive in (4.16). Inserting (4.19) and (4.20) into (4.18) gives

$$|L(p)| < \sum_{n=0}^{(g-r)/2} \tilde{a}_n p^{2n}(p^{-2+2\epsilon})^{n+g/2} \int \prod_{k=1}^{D-1} dX_k\, \exp\left(-A\frac{p^2}{R^2}\right) \exp\left(-S_{D-1}(X)\right). \tag{4.21}$$

If $0 < R < 1$, the behaviour is exponential in p. For $R > 1$, a bound on the angular part of the integral has been calculated in [9], and we can insert the result directly into (4.21) to obtain

$$|L(p)| < \sum_{n=0}^{(g-r)/2} \tilde{a}_n p^{-g+\epsilon(2n+g)} \int_1^\infty dR \exp\left(-A\frac{p^2}{R^2}\right) R^{-k_c(D-1)-1+\epsilon'}, \tag{4.22}$$

where ϵ' is another arbitrarily small positive number, and

$$k_c(D) = 2rD - D - 4r - \delta_{D,3}\delta_{r,2}. \tag{4.23}$$

Finally then, we obtain

$$|L(p)| < \text{const } p^{-k_c(D-1)-g+\epsilon''}\,, \tag{4.24}$$

where ϵ'' is a new arbitrarily small but positive number.

These arguments can be extended to some correlation functions of Polyakov lines. Consider

$$\begin{aligned} L(p,q) &= \langle \mathrm{Tr}\,\exp(ipX_1)\mathrm{Tr}\,\exp(iqX_1)\rangle \\ &= N\int d\lambda_D \Delta_G^2(\lambda_D) dX_k \sum_h \exp(ip\lambda_D\cdot h)\sum_{h'}\exp(iq\lambda_D\cdot h')\times \\ &\qquad \times\exp\left(-\sum_{\alpha>0}\frac{(\lambda_D\cdot\alpha)^2}{|\alpha|^2}\sum_{k=1}^{D-1}\left|\overline{X}_k^\alpha\right|^2 - S_{D-1}\right). \end{aligned} \tag{4.25}$$

Clearly the terms with $h = h'$ satisfy the same bound as $L(p)$ with p replaced by $p+q$. After completing the square, the cross terms become

$$\begin{aligned} L(p) = \int \frac{d\underline{\nu}}{\det(Q)^{\frac{1}{2}}}\prod_{k=1}^{D-1}dX_k\prod_{\alpha\in\Delta_+}\left(\left(Q^{-\frac{1}{2}}\nu + iQ^{-1}\sigma\right)\cdot\alpha\right)^2\exp(-\sigma\cdot Q^{-1}\cdot\sigma)\times \\ \times\exp\left(-\nu\cdot\nu+\sum_{j,k=1}^{D-1}\mathrm{Tr}\,[X_j,X_k]^2\right), \end{aligned} \tag{4.26}$$

where, for example (and specializing to su(3)),

$$\sigma = p\omega_1 - q\omega_2\,. \tag{4.27}$$

Introducing an arbitrary constant $0<\beta<1$ we find that

$$\begin{aligned} \sigma^T Q^{-1}\sigma = \frac{4}{9(T_1T_2+T_2T_3+T_3T_1)}\Big(\beta|\sigma|^2(T_1+T_2+T_3) + T_1((2p-q)^2-\beta|\sigma|^2) + \\ + T_3((p-2q)^2-\beta|\sigma|^2) + T_2((p+q)^2-\beta|\sigma|^2)\Big)\,. \end{aligned} \tag{4.28}$$

It follows that, in the region

$$\begin{aligned} (2p-q)^2-\beta|\sigma|^2 &= (2p-q)^2-\beta(p^2+pq+q^2) > 0\,, \\ (2q-p)^2-\beta|\sigma|^2 &= (2q-p)^2-\beta(p^2+pq+q^2) > 0\,, \end{aligned} \tag{4.29}$$

we have

$$\sigma^T Q^{-1}\sigma \ge \beta|\sigma|^2(\mu_1+\mu_2) \ge const\frac{\beta|\sigma|^2}{R^2}\,, \tag{4.30}$$

where $\mu_{1,2}$ are the eigenvalues of Q^{-1} so that

$$\left|x^T Q^{-1} y\right| \le |x||y|\frac{\sigma^T Q^{-1}\sigma}{\beta|\sigma|^2} \tag{4.31}$$

and

$$|L(p)| < \sum_{n=0}^{(g-r)/2} \tilde{a}'_n |\sigma|^{-g} (\sigma^T Q^{-1} \sigma)^{n+g/2} \int \prod_{k=1}^{D-1} dX_k \, \exp(-\sigma^T Q^{-1} \sigma) \exp(-S_{D-1}(X)) \, , \tag{4.32}$$

where the constants $\tilde{a}'_n$ depend on β. Again, unless $\sigma^T Q^{-1} \sigma < |\sigma|^{2\epsilon}$ we get exponential behaviour so proceeding as for $L(p)$ we obtain

$$L(p,q) < const|\sigma|^{-k_c(D-1)-g+\epsilon'''} \, . \tag{4.33}$$

Keeping β small but fixed, we see that in the regions not satisfying (4.29) σ is almost proportional to a root. Since the roots are weights for the adjoint representation there is screening and it is easy to see that we just get a constant upper bound instead of (4.33).

4.2.2 su($N > 3$)

We now need to generalise the formula 4.16 for $\omega^T Q^{-1} \omega$ to SU(N). First we show that for Q defined as in (4.12),

$$\det Q = \frac{r+1}{2^r} \sum_{\{\alpha_1,\ldots,\alpha_r\} \in \mathcal{S}(r)} T_{\alpha_1} \cdots T_{\alpha_r} \, , \tag{4.34}$$

where $\mathcal{S}(n)$ is the set of all n-tuples of positive roots α which are linearly independent. To see this, we use the fact that the determinant is a homogeneous function of the T_α of degree r. To find the coefficient of $T_{\alpha_1} \cdots T_{\alpha_r}$ for any choice of the $\{\alpha_1 \cdots \alpha_r\}$, set $T_{\alpha_i} = 1$, $i = 1, \cdots, r$ and $T_\alpha = 0$ otherwise. Then Q becomes

$$Q_{ij} = \frac{1}{2} \sum_{k=1}^{r} \alpha^i_k \alpha^j_k \tag{4.35}$$

and

$$\det Q = 2^{-r} \det(\alpha_1, \ldots, \alpha_r) \det(\alpha_1, \ldots, \alpha_r)^T \, , \tag{4.36}$$

where $(\alpha_1, \cdots, \alpha_r)$ is the matrix whose columns are the α_i. If $\{\alpha_1, \cdots, \alpha_r\}$ are linearly dependent then $\det Q = 0$. If $\{\alpha_1, \cdots, \alpha_r\}$ are linearly independent, then, since for su(N) any root is a Weyl transformation of any other root and any positive root can be written in terms of simple roots as

$$\alpha = s_j + s_{j+1} + \cdots + s_{k-1} + s_k \tag{4.37}$$

with $1 \leq j \leq k \leq r$, we can use a series of Weyl transformations and column operations to reduce

$$(\alpha_1, \cdots, \alpha_r) \to (\pm s_1, \cdots, \pm s_r) \, . \tag{4.38}$$

Any minus signs can be dropped since we are only after the square of the determinant. The product of $(s_1, \cdots, s_r)$ and its transpose is just the Cartan matrix whose determinant is $r+1$ so

$$\det Q = \frac{r+1}{2^r} \, , \tag{4.39}$$

which completes the proof of (4.34).

Now we show that

$$\omega_1^T Q^{-1} \omega_1 = 2\frac{\sum_{s=\{\alpha_1,\cdots,\alpha_{r-1}\}\in\mathcal{S}(r-1)} \left(\frac{n_s}{N}\right)^2 T_{\alpha_1}\cdots T_{\alpha_{r-1}}}{\sum_{\{\alpha_1,\cdots,\alpha_r\}\in\mathcal{S}(r)} T_{\alpha_1}\cdots T_{\alpha_r}}\,, \tag{4.40}$$

where the n_s are non-zero integers. This will allow us to use the same form of bound (4.17) as for SU(3).

It is helpful to use a basis in which

$$\omega_1 = \sqrt{\frac{r}{r+1}}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}. \tag{4.41}$$

In this basis

$$\omega_1^T Q^{-1} \omega_1 = \frac{r}{r+1} {Q^\perp}_{1,1} / \det Q\,, \tag{4.42}$$

where $Q^\perp$ is the matrix of cofactors. Our task then is to compute the cofactor ${Q^\perp}_{1,1}$, and as this is a homogeneous function of the T_α of degree $r-1$, we can set $T_{\beta_{(1)}} = \cdots = T_{\beta_{(r-1)}} = 1$ and $T_\alpha = 0$ otherwise, as before. Then

$${Q^\perp}_{1,1} = 2^{1-r}\det\overline{Q}\,, \tag{4.43}$$

where the matrix $\overline{Q}$ is given by

$$\overline{Q} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \sum_{n=1}^{r-1}\tilde\beta_{(n)}\tilde\beta_{(n)}^T & \\ 0 & & & \end{pmatrix} \tag{4.44}$$

with

$$\begin{aligned} \tilde\beta^1_{(n)} &= 0 \\ \tilde\beta^i_{(n)} &= \beta^i_{(n)} \qquad i \neq 1\,. \end{aligned} \tag{4.45}$$

Proceeding as before we note that

$$\det\overline{Q} = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \tilde\beta_{(1)} & \cdots & \tilde\beta_{(r-1)} \\ 0 & & & \end{pmatrix}^2 . \tag{4.46}$$

So, using linear column operations, we can restore the $\tilde\beta_{(i)}$ to $\beta_{(i)}$ to get

$$\det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \tilde\beta_{(1)} & \cdots & \tilde\beta_{(r-1)} \end{pmatrix} = \sqrt{\frac{r+1}{r}}\det(\omega_1, \beta_{(1)}, \cdots, \beta_{(r-1)})\,. \tag{4.47}$$

If the $\beta_{(i)}$ are linearly dependent, then $\det\overline{Q} = 0$. Using the same procedure as before, we obtain

$$\det(\omega_1, \beta_{(1)}, \cdots, \beta_{(r-1)}) = \det(w\omega_1, s_{(1)}, \cdots, s_{(r-1)})\,, \tag{4.48}$$

where w is a Weyl transformation, and the $s_{(i)}$ are $r-1$ of the simple roots. We note that this determinant can be zero in some representations of the algebra. For example, in the adjoint representation in which the weights are roots. However, we now show that it is never zero in the fundamental representation. First, since any Weyl transformation is the product of Weyl reflections in roots, we have

$$w\omega_1 = \omega_1 - 2\sum_{i=1}^{r} p_i s_i \tag{4.49}$$

where the p_i are integers. In terms of the simple roots, ω_1 is given by

$$\omega_1 = \frac{1}{r+1}\left[r s_1 + (r-1)s_2 + \cdots + s_r\right] \tag{4.50}$$

so we find

$$\begin{aligned}\det(w\omega_1, s_{(1)}, \cdots, s_{(r-1)}) &= \det\left(\frac{q}{r+1}s_{(r)} - 2n s_{(r)}, s_{(1)}, \cdots, s_{(r-1)}\right)\\ &= \pm\frac{n_s}{r+1}\det(s_1, \cdots, s_r),\end{aligned} \tag{4.51}$$

where q is an integer between 1 and r, n is one of the p_i, also an integer, and n_s is a non-zero integer. Then we have

$$\det \overline{Q} = \frac{n_s^2}{r} \tag{4.52}$$

and, together with the expression (4.34) for $\det Q$, this completes the result (4.40).

The upper bound for $\mathrm{SU}(N)$ now follows in a manner identical to the proof of (4.40). The cofactors of Q take the general form

$$Q_{ij}^{\perp} = \sum_{s=(\alpha_1\cdots\alpha_{r-1})\in\mathcal{S}(r-1)} C_s\, T_{\alpha_1}\cdots T_{\alpha_{r-1}}\,, \tag{4.53}$$

where the C_s are positive constants, so that we have again the result (4.17)

$$x^T Q^{-1} y < \mathrm{const}\,|x||y|\omega^T Q^{-1}\omega\,. \tag{4.54}$$

The remainder of the argument follows without modification from the su(3) case, and we have again

$$|L(p)| < \mathrm{const}\, p^{-k_c(D-1)-g+\epsilon}\,, \tag{4.55}$$

where the function k_c is given in 4.23, and ϵ is arbitrarily small.

5. Conclusions

For the bosonic su(2) model the exact calculation shows that asymptotically $L(p) \sim p^{2-D}$ while the upper bound (3.14) demonstrates that this behaviour arises from the flat directions in the action. For larger gauge groups there is unfortunately no exact calculation but again the flat directions lead to a power law bound on $L(p)$ at large p (4.55). In the process of proving the bound we throw away the alternating signs coming from the Vandermonde in (4.11); it is possible that in an exact calculation these would lead to the cancellation of the leading power of p. However, as we remarked previously, this does not happen in su(2) and so there is no reason to suppose that it happens for larger groups. We conclude that $L(p)$ has the asymptotic behaviour

$$L(p) \sim \text{const}\, p^{-k_c(D-1)-g} \tag{5.1}$$

at large p. Our proof of this bound uses certain properties that are special to the su(N) Lie algebra but, just as the convergence proofs go through for all gauge groups, we expect that very similar results for $L(p)$ will hold for all the compact Lie groups. This power law property is different from the exponential asymptotic behaviour given by the mean field approximation [33]. The reason is that the mean-field saddle point computation misses the dominant region of integration at large p which is essentially the end-point of the integration domain rather than the saddle point.

Although the asymptotic behaviour of the supersymmetric su(2) model is qualitatively different, being exponential rather than power law, it is not clear whether this extends to supersymmetric su(N). In the su(2) model the power law is suppressed whenever there are fermions in the adjoint representation so this is a feature of su(2) rather than of supersymmetry. The asymptotic behaviour of the supersymmetric su(N) models is an open question.

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JHEP09(2003)023

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